0521
S. Ariki

1. $A = (a_{ij})_{i,j\in I}$: symmetrizable Cartan matrix

$(P, \Pi = \{\alpha_i\}_{i\in I}, \Pi^\vee = \{h_i\}_{i\in I})$

(i) P : free $\mathbb{Z}$-module of finite rank equipped with a $\mathbb{Q}$-valued symmetric bilinear form $(\ ,\ )$

(ii) Π is a set of linearly independent vectors in P

$\Pi^\vee$ 〃 $P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$

(iii) $\langle \alpha_j, h_i\rangle = a_{ij}$ $\quad \langle\ ,\ \rangle : P \times P^\vee \to \mathbb{Z}$

$$\langle \Lambda, h_i\rangle = \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$$

(iv) $\dfrac{(\alpha_i, \alpha_i)}{2} \in \mathbb{Z}_{>0}$

$\rightsquigarrow$ quantum algebra $U_v(\mathfrak{g}(A)) =: U_v$

generated by $\{E_i\}_{i\in I}, \{F_i\}_{i\in I}, \{v^{h}\}_{h\in P^\vee}$

subject to the well-know relation

Def. $\mu, \mu' \in P$ $\qquad {}_\mu(U_v)_{\mu'} = U_v \Big/ \sum_{h\in P^\vee} (v^{h} - v^{\langle \mu, h\rangle}) U_v + \sum U_v (v^{h} - v^{\langle \mu', h\rangle})$

set $\dot{U}_v = \bigoplus_{\mu,\mu'\in P} {}_\mu(U_v)_{\mu'}$

associative alg. (without unit)

mult. ${}_\lambda(U_v)_\mu \times {}_{\mu'}(U_v)_\nu \longrightarrow 0$ if $\mu \neq \mu'$

〃 $\longrightarrow {}_\lambda(U_v)_\nu$ if $\mu = \mu'$

$\bar{a}, \bar{b} \mapsto \overline{ab}$ well-defined

modified quantum algebra introduced by Lusztig

Rem
(i) $\mu(\mathcal{U}_v)_{\mu'} = \bigoplus_{\alpha-\beta=\mu-\mu'} \mathcal{U}_\alpha^{+} \cdot \mathcal{U}_{-\beta}^{-}$

root spaces

$a_\mu := 1 \in \mu(\mathcal{U}_v)_\mu$ Then $\dot{\mathcal{U}}_v = \bigoplus_{\mu \in P} \mathcal{U}_v a_\mu$

(ii) M : $\mathcal{U}_v$-module which admits a weight decomposition $M = \oplus M_\mu$

Then
$$\begin{aligned} \mu(\mathcal{U}_v)_{\mu'} \times M_\nu &\longrightarrow 0 && \text{if } \mu' \neq \nu \\ &\longrightarrow M_\mu && \text{if } \mu' = \nu \\ (\bar{a}, m) &\longmapsto am \end{aligned}$$
defines a $\dot{\mathcal{U}}_v$-module structure.

For a dominant integral weight $\xi, \zeta \in P^+$

$V(\xi) \supset L(\xi) = \sum R \tilde{f}_{i_1} \cdots \tilde{f}_{i_N} u_\xi$, $B(\xi) \subseteq L(\xi)/vL(\xi)$

h.w module

$V(-\zeta) \supset L(-\zeta) = \sum R \tilde{e}_{i_1} \cdots \tilde{e}_{i_N} u_{-\zeta}$, $B(-\zeta) \subset L(-\zeta)/vL(-\zeta)$

lowest wt module

$R = \{ \frac{f(v)}{g(v)} \mid g(0) \neq 0 \}$ $\tilde{e}_i, \tilde{f}_i$: Kashiwara operators

(crystal bases)

canonical bases $\{G(b) \mid b \in B(\xi)\}$, $\{G(b) \mid b \in B(-\zeta)\}$

which are characterised by

(i) $G(b) + vL(\xi) = b$, $G(b) + vL(-\zeta) = b$

(ii) $\overline{G(b)} = G(b)$ where $\overline{Pu_\xi} := \bar{P} u_\xi$ (or $\overline{Pu_{-\zeta}} = \bar{P} u_{-\zeta}$)

where $\overline{E_i} = E_i$, $\overline{F_i} = F_i$, $\overline{v^{h}} = v^{-h}$, $\overline{v} = v^{-1}$

coproduct
$$\begin{cases} \Delta(v^{h}) = v^{h} \otimes v^{h} \\ \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i \\ \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i \end{cases}$$

$V(\xi) \otimes V(-\zeta)$: U_v-module, hence $\dot{U}_v$-module
$$\parallel$$
$U_v(u_\xi \otimes u_{-\zeta})$

We can define bar operation on $V(\xi) \otimes V(-\zeta) = V(\xi, -\zeta)$
by $\overline{P(u_\xi \otimes u_{-\zeta})} = \overline{P}\, u_\xi \otimes u_\zeta$

$L(\xi, -\zeta) := L(\xi) \otimes_R L(-\zeta)$, $B(\xi, -\zeta) = B(\xi) \otimes B(-\zeta)$

Then we have elements $G(b_1 \otimes b_2)$ characterised by

(i) $G(b_1 \otimes b_2) + vL(\xi, -\zeta) = b_1 \otimes b_2$

(ii) $\overline{G(b_1 \otimes b_2)} = G(b_1 \otimes b_2)$

$\{G(b_1 \otimes b_2)\}_{b_1 \otimes b_2 \in B(\xi, -\zeta)}$ is the canonical base of $V(\xi, -\zeta)$
(Lusztig)

Recall we have strict embeddings of crystals

$B(\xi) \subset B(\infty) \otimes T_\xi$

$B(-\zeta) \subset T_{-\zeta} \otimes B(-\infty)$

$$\dot{B} = \bigsqcup_{\mu \in P} B(\infty) \otimes T_\mu \otimes B(-\infty) = \varinjlim_{\xi, \zeta \to \infty} B(\xi, -\zeta)$$

__Define__ The canonical basis / global bases $\dot{\mathbb{B}} \subset \dot{U}_v$ is the $\mathbb{Q}(v)$-basis of $\dot{U}_v$ characterized by the property that

$$\dot{U}_v \supset U_v a_\mu \longrightarrow V(\zeta, -\xi) \qquad \zeta - \xi = \mu$$

$$a_\mu \longmapsto u_\zeta \otimes u_{-\xi}$$

sends $G(b_1 \otimes t_\mu \otimes b_2)$ to $G(\underbrace{b_1 \otimes t_\zeta}_{\in B(\zeta)}, \underbrace{t_{-\xi} \otimes b_2}_{\in B(-\xi)})$

$(T_\mu = T_\zeta \otimes T_{-\xi})$

or 0.

__Note__ $G(b_1 \otimes t_\mu \otimes 1) = G(\underset{\in U_v^-}{b_1}) a_\mu$

$G(1 \otimes t_\mu \otimes b_2) = G(\underset{\in U_v^+}{b_2}) a_\mu$

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2. cyclotomic Hecke algebra

Broué - Malle - Rouquier by using KZ functor for any complex reflection groups

braid group $= \pi_1$ (complement of hyperplane arrangement)

$\rightsquigarrow$ cyclotomic Hecke alg is quotient

For $G(m,1,n) = \mathbb{Z}/m\mathbb{Z} \wr \mathfrak{S}_n$, it was also introduced by generators & relations — Ariki-Koike

quotient of (extended) affine Hecke algebras (Cherednik)

We work over $\mathbb{C}$.

Def. cyclotomic Hecke algebra ass. with $G(m,1,n)$ is the $\mathbb{C}$-alg. family

$$\mathcal{H}_n(v_1,\dots,v_m,q) \qquad (v_1,\dots,v_m,q\in\mathbb{C}^*)$$

with generators $T_0,\dots,T_{n-1}$ and

$$(T_0-v_1)\cdots(T_0-v_m)=0$$
$$(T_i-q)(T_i+1)=0 \qquad 1\le i<n$$
$$(T_0T_1)^2=(T_1T_0)^2$$
$$T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1} \qquad (1\le i<n-1)$$
$$T_iT_j=T_jT_i \qquad (j\ge i+2)$$

By a theorem by Dipper-Mathas, we may assume

$$v_i=q^{\gamma_i}, \quad q=\sqrt[e]{1} \qquad 2\le e\le\infty,\ \gamma_i\in\mathbb{Z}/e\mathbb{Z}$$

without loosing (much) information $\qquad (\gamma_i\in\mathbb{Z}$ if $e=\infty)$

$$\mathfrak{g}=\mathfrak{g}(A^{(1)}_{e-1})=\mathbb{C}[t,t^{-1}]\otimes\mathfrak{sl}_e\oplus\mathbb{C}c\oplus\mathbb{C}d$$

Λ_i : fundamental weight $\quad \langle\Lambda_i,d\rangle=0,\ \langle\Lambda_i,h_j\rangle=\delta_{ij}$

$\Lambda=\Lambda_{\gamma_1}+\cdots+\Lambda_{\gamma_m}$

We write $\mathcal{H}_n^{\Lambda}(q)$ instead of $\mathcal{H}_n(v_1,\dots,v_m,q)$.

Def. a $\mathbb{C}$-linear category $\mathcal{Q}_{\mathfrak{g}}$

objects $=P$

morphisms $\qquad \mathrm{Hom}(\mu,\nu)=\dfrac{\{\text{linear combinations of paths } \mu\to\mu\pm\alpha_i\to\cdots\to\nu\}}{\text{quantum alg. relations}}$

$$\begin{tikzcd} \mu \arrow[r, "E_{i,\mu}"] \arrow[d, "F_{j,\mu}"'] & \mu+\alpha_i \arrow[d, "F_{j,\mu+\alpha_i}"] \\ \mu-\alpha_j \arrow[r, "E_{i,\mu-\alpha_j}"'] & \mu+\alpha_i-\alpha_j \end{tikzcd}$$

$$F_{j,\mu+\alpha_i}E_{i,\mu} = E_{i,\mu-\alpha_j}F_{j,\mu}$$
if $i \neq j$

$$E_{i,\mu-\alpha_i}F_{i,\mu} - F_{i,\mu+\alpha_i}E_{i,\mu} = \frac{v^{\langle \mu, h_i\rangle} - v^{-\langle h_i, \mu\rangle}}{v - v^{-1}}$$

and Serre relations

A $\mathbb{C}$-linear morphism $Q_g \longrightarrow Vect_{\mathbb{C}}$ … finite dim

is called a $\dot{U}_v$-module

<u>Def.</u> M : $\dot{U}_v$-module

(abelian) categorification of M is an assignment

abelian

$\mu \in P \rightsquigarrow$ additive category $\mathcal{M}_\mu$

$\varphi \in \mathrm{Hom}(\mu, \nu) \rightsquigarrow$ exact functor $F_\varphi : \mathcal{M}_\mu \to \mathcal{M}_\nu$

where $\nu = \mu \pm \alpha_i$

s.t. $K_0^{split}(\mathcal{M}_\mu) \otimes_{\mathbb{Z}} \mathbb{C} = M_\mu$

$K_0(\quad)$

$[F_\varphi] = M(\varphi) : M_\mu \to M_\nu$

Return to $\mathcal{H}_n^\wedge(q)$

Define the Jucys-Murphy elements

$L_1, \cdots, L_n \in \mathcal{H}_n^\wedge(q)$ by

$$L_1 = T_0$$
$$L_{i+1} = q^{-1} T_i L_i T_i \quad (1 \leq i < n)$$

$\Rightarrow L_1, \cdots, L_n$ pairwise commute

symmetric poly. in $L_1, \cdots, L_n$ is central in $\mathcal{H}$

Assumption

$v_1, \dots, v_n \in \mathfrak{g}^{\mathbb{Z}}$

M: simple $\Rightarrow \prod_{i=1}^{n}(t - L_i)$ (t: indeterminate)

acts on M by $\prod_{j=1}^{n}(t - \mathfrak{g}^{i_j}) \in \mathbb{C}[t]$ for some $i_1, \dots, i_n$

Define $\mathrm{wt}(M) = \Lambda - \sum_{j=1}^{n} \alpha_{i_j}$

Th (Lyle-Mathas)

Two simple module M_1, M_2 belongs to a same block.

$\Longleftrightarrow \mathrm{wt}(M_1) = \mathrm{wt}(M_2)$

Hence block algebras of $\mathcal{R}_n^{\Lambda}(\mathfrak{g})$ are parametrised by a subset $\subset P$

B_μ ($\mu \in P$) block labelled by μ

Th (A + Lyle-Mathas)

$\mu \in P(V(\Lambda)) \rightsquigarrow B_\mu\text{-mod}$ $B_\mu\text{-proj}$

$\notin$ 〃 $\rightsquigarrow 0$

$E_{i,\mu} \in \mathrm{Hom}(\mu, \mu + \alpha_i)$

$\mapsto i\text{-res}.$ (or 0)

$F_{i,\mu} \in \mathrm{Hom}(\mu, \mu - \alpha_i)$

$\mapsto i\text{-ind}$ (or 0)

is an (abelian) categorification of the integrable $\dot{U}_{v=1}(\widehat{\mathfrak{sl}}_e)$-module $V(\Lambda)$

Furthermore the global basis $G(b)$ at $v=1$ maps $[P(\mu)] \in V(\Lambda)$

3. Khovanov - Mazorchuk - Stroppel introduced the following

Def. A $\mathbb{Z}$-alg. with basis $\mathbb{B}$

Assume $\mathbb{B}$ is positive, i.e. the structure constant $\in \mathbb{Z}_{\geq 0}$.

V : A-module

A weak abelian categorification of $(A, \mathbb{B}, V)$ is an abelian category $\mathcal{D}$ together with exact functors $\{F_b : \mathcal{D} \to \mathcal{D}\}$ $(b \in \mathbb{B})$ s.t.

(1) $K_0(\mathcal{D}) = V$

(2) $[F_b]$ = action of b on V.

(3) $F_b F_{b'} = \bigoplus_{b'' \in B} F_{b''}^{\oplus c_{bb'}^{b''}}$ $\qquad c_{bb'}^{b''}$ = str. const.

$\mathbb{C}$-alg. A and relax (1) to $K_0(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{C} = V$

Lemma $A = U^-$, canonical basis $B \subset U^-$

$\Rightarrow$ $\mathcal{D} = \bigoplus_{n \geq 0} \mathcal{H}_n^{\wedge}$-mod is a weak abelian categorification of $(U^-, B, V(\Lambda))$

Conj. [KMS] We can define F_b for $G(b) \in \dot{B}$ s.t.

(a) the same $\mathcal{D} = \bigoplus B_\mu$-mod is a weak cat. of $(\dot{U}, \dot{\mathbb{B}}, V(\Lambda))$

(b) F_b are ind. direct summand of compositions of i-ind & i-res.

cf. Rickard complex -- T in Rouquier's categorification